considered, pitch motions have been reduced by a factor of two or more, and attitude oscillations limited to 0.5–1.0 about fixed biases for all axes. The fixed biases are known, however, through the attitude determination facility afforded by the simulation. Compensation can be effective, without magnet readjustments, for periods ranging from several weeks to months.

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A Magnetic Attitude Control System for an Axisymmetric Spinning Spacecraft

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This paper describes a theoretical study of a continuous magnetic attitude control system which points the spacecraft spin axis normal to a highly eccentric orbit plane and maintains nearly constant spin speed. A Kalman filter is used to process sampled roll error measurements and produce estimates of yaw error and attitude rates which enable the generation of minimum energy control, active nutation damping, and fast transient response for removing pointing errors. An algebraic solution of a pointing control law which uses the error state estimate to decrease energy requirements is developed. Mechanization of both pointing and spin-speed control is obtained with three logic modes based upon spin-speed error. A Lyapunov-function-generating technique is used with averaging techniques to demonstrate stability of the resulting system operating in a fluctuating magnetic field. Dynamic performance characteristics of the controller are compared to those of a simpler system without the filter.

Introduction

A SPACECRAFT which is spin-stabilized about its maximum axis of inertia is often used for Earth orbital applications because of its inherent ability to hold a fixed spin-axis orientation. The degree of accuracy to which a spin axis can be pointed is highly dependent upon the vehicle attitude sensors available and the orbital elements.

This paper is concerned with a control system which will accurately point the spin axis of such a satellite normal to the plane of a highly eccentric orbit while sustaining the spin speed. The control system has direct application to a class of drag-free geodesy satellites,[1] but it can be used for most spining satellites requiring such orbit-plane orientation. In particular, magnetic actuation of this control is investigated. The desired torque is produced by creating a dipole on the spacecraft which interacts with the Earth's magnetic field. It is assumed that the only attitude error measurements come from horizon sensors.

Much of the previous work on magnetic controllers is related to the above control problem. Wheeler[2] investigated the use of a single dipole aligned with the spin axis for pointing control. Vrablik et al.[3] discussed using electromagnetic coils with axes perpendicular to the spin axis of the LES 4 satellite for maintaining the spin axis nearly normal to a circular, equatorial orbit. Fischell[4] recognized that magnetic control could be used for regulating the spin speed of the satellite. Sonnabend[5] described a controller for keeping the spin axis of a dual-spin satellite normal to the plane of a circular orbit. The first operational magnetically controlled spinning spacecraft, the TIROS satellites, were discussed by Hecht and Manger.[6] This paper combines and extends these results to a system with limited attitude determination capability operating in an eccentric orbit.

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Coordinate Systems and Equations of Motion

Figure 1 illustrates the local reference frame of the spacecraft which has unit vector $\hat{x}_l$ oriented outward along the radius vector from the geocenter. The unit vector $\hat{z}_l$ is normal to the orbit plane and positively oriented toward the northern hemisphere, and $\hat{y}_l$ completes the right hand set. The angles i, Ω, ω_p, and f are the inclination, right ascension,

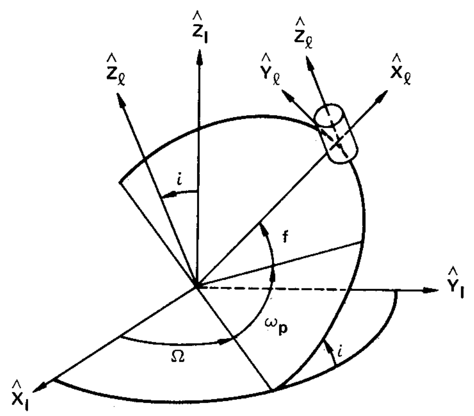

Fig. 1 Orientation of the local l reference frame.

argument of perigee, and true anomaly, respectively. The vehicle spin axis should point along the $\hat{z}_l$ axis.

Figure 2 depicts the transformation from the reference axes to the spacecraft body axes by successive rotations about the $\hat{x}_l$, $\hat{y}_p$, and $\hat{z}_b$ axes, through angles ϕ, θ, and ψ. It is assumed that the body axes labeled b are principal axes about which control torques are applied. Also, the vehicle has inertial symmetry about the nominal spin axis $\hat{z}_b$ which is the axis of maximum moment of inertia. The angles ϕ and θ (referred to here as yaw and roll) are assumed to be small. The spin rate about $\hat{z}_b$ is much larger than the rates ω_x and ω_y about $\hat{x}_b$ and $\hat{y}_b$, so a set of linearized equations can be used to describe the vehicle's motion.

The attitude rates about the nonspinning $\hat{x}_p,\hat{y}_p$ axes are defined as

$$\begin{aligned}\alpha_x &\triangleq \omega_x \cos\psi - \omega_y \sin\psi \\ \alpha_y &\triangleq \omega_x \sin\psi + \omega_y \cos\psi\end{aligned} \tag{1}$$

and the orbital rate is

$$\dot{\sigma} \triangleq \dot{f} + \dot{\omega}_p \tag{2}$$

Also, D is defined to equal $(I_{zz}/I_{xx})\dot{\psi}_0$, where I_{zz} and I_{xx} are the spin-axis and lateral moments of inertia of the vehicle, and $\dot{\psi}_0$ is the nominal spin rate. (It will be seen that nearly constant spin rate can be maintained by control action). The parameter d_p^{-1} is the time constant of a mechanical nutation damper used in the "energy-sink" method[7] to describe the effect of the damper on angular rates.

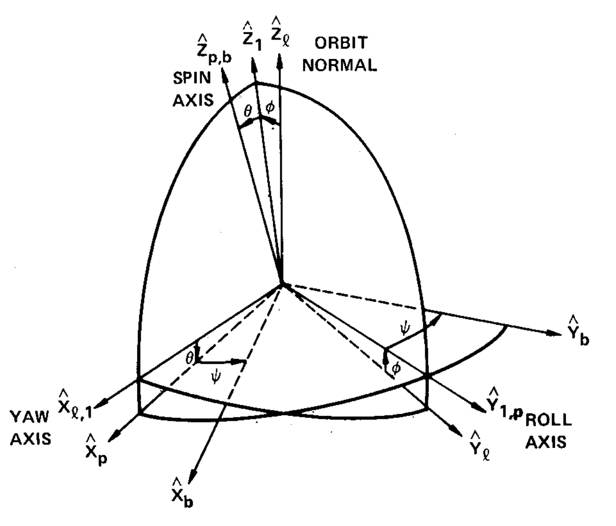

Fig. 2 Transformation from a reference frame l to a body fixed frame b.

If Euler's equations and the kinematic orbital relationships between ϕ and θ are employed, the linearized equations of attitude motion for the vehicle become[8]

$$\begin{bmatrix}\dot{\alpha}_x \\ \dot{\alpha}_y \\ \dot{\phi} \\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}-d_p & -D & 0 & 0 \\ D & -d_p & 0 & 0 \\ 1 & 0 & 0 & \dot{\sigma} \\ 0 & 1 & -\dot{\sigma} & 0\end{bmatrix}\begin{bmatrix}\alpha_x \\ \alpha_y \\ \phi \\ \theta\end{bmatrix} + \begin{bmatrix}1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}T_x \\ T_y\end{bmatrix} \tag{3}$$

The torque components T_x and T_y are the sum of environmental disturbances and the applied control torque divided by I_{xx}. Note that the assumption of inertial symmetry about the spin axis produced state equations with complex symmetry[9] [i.e., Eqs. (3) can be reduced to two complex equations]. This property is used to advantage later in deriving the control law and analyzing the system stability.

State Estimation

It is assumed that sampled measurements θ_s of the roll error are provided by two infrared bolometers mounted with their optical axes in a "vee" configuration in a plane containing the nominal spin axis. As the satellite spins, the optical axes sweep out two cones in space. Each sensor periodically responds to the radiance change existing between outer space and the Earth. The sensor outputs have different pulse widths (Fig. 3) which are functions of the roll error. These pulse trains can be utilized to measure the vehicle's spin speed and current orbital rate $\dot{\sigma}$, since the average pulse width varies with altitude and spin rate. No other attitude measuring device is considered here.

For attitude control system efficiency, it is necessary to determine the yaw angle ϕ. If active damping is to be provided, the values of the vehicle attitude rates must also be known. The angle ϕ and the rates α_x and α_y can be estimated from the roll-error measurements due to the orbital cross-coupling which exists between ϕ and θ. A Kalman filter, now described, processes the sensor measurements to account for random disturbance torques and measurement noise.

If the system state, the control torque, and the disturbance torque are designated by the vectors $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{v}$, respectively, the differential equations (3) can be placed in the form

$$\dot{\mathbf{x}} = \mathbf{Fx} + \mathbf{Gu} + \mathbf{Gv} \tag{4}$$

The disturbance torques consist mainly of constants and oscillatory terms in the spacecraft's reference frame. If estimates of any of the disturbance torque components are available, a control torque can be generated to counter that component's effect and improve the control accuracy. However, an important component—a constant torque about the yaw axis (such as from aerodynamic drag)—is not observable from roll-error measurements of a symmetric vehicle. This type of torque is chiefly responsible for accuracy limitations of the control system presented here.

If the roll error is corrupted by measurement noise w, the sampled measurement θ_s can be represented by

$$\theta_s = [0 \quad 0 \quad 0 \quad 1]\mathbf{x} + w = \mathbf{Hx} + w \tag{5}$$

at the measurement time. For explanatory simplicity, the disturbance torque $\mathbf{v}$ and the measurement noise w are treated here as uncorrelated white noise with covariances

$$\begin{aligned}E\{\mathbf{v}(t)\mathbf{v}^T(\tau)\} &= \mathbf{R}_1\delta(t - \tau) \\ E\{w_i w_k\} &= R_2\delta_{ik}\end{aligned} \tag{6}$$

where $\mathbf{R}_1$ is a positive definite matrix and R_2 is a positive number.

Let us define $\mathbf{\Sigma}(t)$ as the covariance matrix of the error $(\mathbf{x} - \mathbf{x}_e)$, where $\mathbf{x}_e$ is the estimate of state $\mathbf{x}$. With a small amount of logic circuitry on the spacecraft, one can implement a continuous Kalman filter which uses sampled inputs from the horizon sensors. That is, between samples, the equations

$$\dot{\mathbf{x}}_e = \mathbf{F}\mathbf{x}_e + \mathbf{G}\mathbf{u} \qquad (7)$$

$$\dot{\mathbf{\Sigma}} = \mathbf{F}\mathbf{\Sigma} + \mathbf{\Sigma}\mathbf{F}^T + \mathbf{G}\mathbf{R}_1\mathbf{G}^T \qquad (8)$$

are integrated to produce the estimate $\mathbf{x}_e$. At the time of measuring the roll error θ_s, the state estimate is updated by the equation

$$\mathbf{x}_e(t+) = \mathbf{x}_e(t-) + \mathbf{K}(\theta_s - \theta_e) \qquad (9)$$

where the gain matrix $\mathbf{K}$ is

$$\mathbf{K} = \mathbf{\Sigma}(t-)\mathbf{H}^T[\mathbf{H}\mathbf{\Sigma}(t-)\mathbf{H}^T + R_2]^{-1} \qquad (10)$$

The covariance is updated by

$$\mathbf{\Sigma}(t+) = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{\Sigma}(t-) \qquad (11)$$

Here, $\mathbf{I}$ is the identity matrix and $(t-)$ and $(t+)$ indicate the instants before and after the sampling point.

If the random variables $\mathbf{v}$ and w have stationary statistics and the sampling interval is constant, then $\mathbf{K}$ reaches an approximately steady-state value. With an orbital eccentricity of 0.7, the components of $\mathbf{K}$ vary less than 2% between apogee and perigee for constant $\mathbf{R}_1$ and R_2. Use of a constant $\mathbf{K}$ is a desirable simplification in implementing the filter; with it, only Eqs. (7) and (9) require onboard mechanization. However, the time variable $\dot{\sigma}$ must be updated regularly in $\mathbf{F}$ of Eq. (7). Also, for good estimation results with a single pair of horizon sensors, the product 2D must not be close to an integer value to insure that Eq. (10) has a well-conditioned solution.

Control Law for Low-Energy Pointing

With the satellite attitude-error state $\mathbf{x}$ known, a feedback control torque can be generated which will drive each state element to zero and maintain it there. If the control torque is to be generated magnetically, a good choice of a control law is one which utilizes minimum electrical energy in addition to providing the desired transient response.

Magnetic control torque is produced by applying voltage to coils of wire fixed on the spacecraft. The magnetic moment and corresponding torque is proportional to the current in each coil. Optimal controller energy is obtained by minimizing the time integral of the sum of the power used by each coil, with constraints being placed upon the attitude error. Therefore, a suitable performance index is quadratic in the applied control $\mathbf{u}$, and is chosen to be (with $t_f - t_0 \to \infty$)

$$J = 0.5 \int_{t_0}^{t_f} (\mathbf{x}^T\mathbf{Q}_1\mathbf{x} + \mathbf{u}^T\mathbf{Q}_2\mathbf{u})dt \qquad (12)$$

Here, $\mathbf{Q}_1$ is a positive semidefinite matrix, and $\mathbf{Q}_2$ is positive definite.

Optimal control of a linear system (4) with performance index (12) requires[10] finding the solution to the Riccati equation

$$\dot{\mathbf{P}} = -\mathbf{P}\mathbf{F} - \mathbf{F}^T\mathbf{P} + \mathbf{P}\mathbf{G}\mathbf{Q}_2^{-1}\mathbf{G}^T\mathbf{P} - \mathbf{Q}_1 \qquad (13)$$

with $\mathbf{P}(t_f) = 0$. The optimal control law is

$$\mathbf{u} = -\mathbf{Q}_2^{-1}\mathbf{G}^T\mathbf{P}\mathbf{x} \qquad (14)$$

This control would be difficult to implement, because the Earth's magnetic field is fluctuating, which causes $\mathbf{G}$ (and therefore $\mathbf{P}$) to be time-varying. A reasonable simplification is to assume the presence of a constant magnetic field (such as the orbital average). Then, with $\mathbf{G}$ constant, a constant-gain suboptimal control law is obtained by solving Eq. (13) for the steady-state solution $\mathbf{P}_\infty$. Because the spinning satellite's state equations, Eqs. (3), have complex symmetry, an algebraic solution to Eqs. (13) can readily be found as outlined in Appendix A.

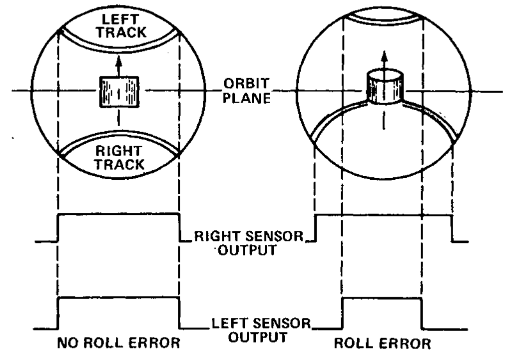

Fig. 3 Horizon sensor output; the Earth is behind the spacecraft.

From Eq. (14), the suboptimal energy control law becomes

$$\begin{aligned} u_x &= -K_v\dot{\phi} - K_{p1}\phi - K_{p2}\theta \\ u_y &= -K_v\dot{\theta} + K_{p2}\phi - K_{p1}\theta \end{aligned} \qquad (15)$$

as applied to minimizing Eq. (A8) of Appendix A. Equations (15) represent the ideal pointing control law used in this paper. This control law is applicable to satellites with and without nutation dampers.

Although the assumption that a constant magnetic field exists is not realistic, mechanization of Eqs. (15) utilizes less energy than a control law which is a function of only the roll error. A comparison of the energy requirements of Eqs. (15) to that of an optimal control law using the time-varying $\mathbf{G}$ in Eqs. (13) and (14) was not made. The effect on attitude stability of using constant gains in a time-varying magnetic field is investigated after the next section.

Magnetic Implementation of the Control Law

Pointing Control

It is usually not necessary to maintain the same degree of control on the spin speed as that required for pointing the spin axis of the satellite. Therefore, concentration is first directed toward achieving magnetic pointing control, with the assumption that the spin speed is nearly its nominal value. The control torque generated by creating a magnetic moment $\mathbf{m}$ on the spacecraft is

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (16)$$

where $\mathbf{B}$ is the local value of the Earth's magnetic field. However, $\mathbf{B}$ is rarely oriented so that a magnetic moment $\mathbf{m}$ can be created which will simultaneously produce all three components of the required $\mathbf{T}$. The magnetic moment components required for implementing a control torque proportional to Eqs. (15) are found from Eq. (16) to be

$$m_x = -K_{Bz}T_{Dy}; \quad m_y = K_{Bz}T_{Dx}; \quad m_z = 0 \qquad (17)$$

Here, T_{Dx} and T_{Dy} are the desired torques [such as those of Eqs. (15)] specified along the nonspinning reference axes, and

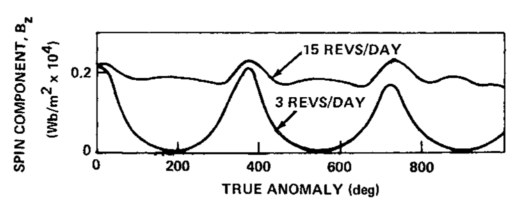

Fig. 4 The spacecraft spin-axis component of the Earth's magnetic field for eccentric orbits with perigee at 300 km, inclination 45°

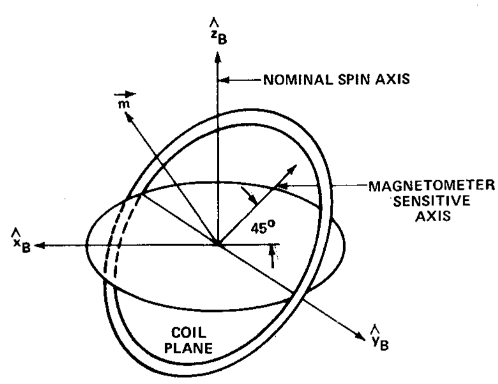

Fig. 5 Geometry of skewed coil in a spinning spacecraft; magnetometer sensitive axis lies in the plane of the coil.

K_{Bz} is a constant approximately equal to the reciprocal of the average value of the spin component of **B**. These components are created by generating spin-modulated moments in the orthogonal body-fixed lateral coils. The resulting torque components are, from Eq. (16),

$$T_x = K_{Bz}B_zT_{Dx}; \quad T_y = K_{Bz}B_zT_{Dy} \tag{18a}$$

$$T_z = -K_{Bz}(B_xT_{Dx} + B_yT_{Dy}) \tag{18b}$$

where B_x, B_y, and B_z are reference frame components of **B**. Under such a scheme, the spin-torque component T_z should average to zero over several orbits if no disturbance torques are present. The pointing control torque components T_x and T_y will equal the desired values T_{Dx} and T_{Dy} in an average sense. To provide the exact values would require replacing the gains K_{Bz} by the factor $(1/B_z)$. This increases the complexity of the required mechanization and is impractical for highly eccentric orbits where B_z can vary over two orders of magnitude (see Fig. 4).

The magnetic field of the Earth is roughly an Earth-centered dipole with its axis inclined about 11° to the Earth spin axis. Thus, for satellite orbits inclined up to 70°, the satellite's spin axis component B_z of the magnetic field always will be positive for small pointing errors, and no problem of sign reversal exists in Eqs. (18).

Partial pointing control can also be obtained with a spin-axis coil, for which a suitable control law is found from Eq. (16) to be

$$m_z = K_{OB}(B_xT_{Dy} - B_yT_{Dx}) \tag{19}$$

where K_{OB} is approximately one over the average squared value of the lateral component of the magnetic field found in the orbit. The advantage of using a spin coil alone is that it produces no unwanted spin-torque component and does not require spin modulation of the applied voltage. However, it offers no means of spin-speed control and produces a torque which is seldom in the direction of the desired pointing control [Eqs. (15)]. Also, it is unsuitable for near equatorial orbits where the magnetic field tends to be parallel to the spin axis.

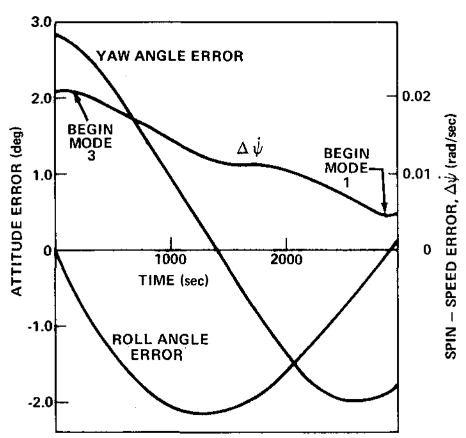

Fig. 6 Typical response of spin speed, roll, and yaw errors during mode 3 control.

The system chosen is one in which the lateral coils and Eqs. (17) are used for pointing control during normal operations (i.e., portions of the orbits where the spin speed is adequately close to the nominal value). During the periods when the lateral coils cannot be used for pointing control because of the spin-speed deviation, the spin coil is used with Eq. (19) for partial pointing control.

Spin Control

The control torque to correct a spin-speed deviation $\Delta\dot{\psi}$ ($= \dot{\psi} - \dot{\psi}_0$, where $\dot{\psi}_0$ is nominal) might be

$$T_{Dz} = -K_z \operatorname{sgn}(\Delta\dot{\psi}) \tag{20}$$

when $|\Delta\dot{\psi}|$ exceeds some deadband value; K_z is another gain constant. There are two ways in which a similar spin control can be magnetically actuated. One occurs as a by-product of the primary pointing control, and is due to Eq. (18b). One can incorporate logic into the control system such that pointing control by Eqs. (17) is actuated only when the resulting spin torque Eq. (18b) is in the direction of Eq. (20). This mechanization is considered the primary means of spin control. The other way is to apply voltage to the lateral coils in such a way that the resulting satellite dipole is normal to the lateral component of the magnetic field. This method requires the components

$$m_x = -K_zB_y \operatorname{sgn}(\Delta\dot{\psi}); \quad m_y = K_zB_x \operatorname{sgn}(\Delta\dot{\psi}) \tag{21}$$

Application of Eqs. (21) will result in a pointing disturbance. Therefore, this second method is used only when spin-speed deviation is large, i.e., $\Delta\dot{\psi}$ approaches a point where serious degradation to pointing control occurs, due to the resulting inaccuracy of the parameter D. The pointing disturbance of Eqs. (21) is minimized by applying this moment over a 180° segment of the orbit centered around the perigee position.[8]

A lateral magnetic field component is required for spin-speed control. Thus, spacecraft with this feature would probably be restricted to orbits with inclinations above 20°.

Combined Control with System Logic

The foregoing control methods are now combined into a single controller with some appropriate logic. Let the constant C_{d1} be the value of $|\Delta\dot{\psi}|$ beyond which one should provide some sort of spin control, and let C_{d2} be the value of $|\Delta\dot{\psi}|$ beyond which spin-control action is mandatory. Define C_p as the value of orbital rate $\dot{\sigma}$, 90° from perigee. Then suitable logic governing the voltages applied to three orthogonal coils is

Mode 1. $(-C_{d1} \leq \Delta\dot{\psi} \leq C_{d1})$: use Eqs. (17).

Mode 2. $(C_{d1} < |\Delta\dot{\psi}| \leq C_{d2})$: a) If $\operatorname{sgn}(T_z) \triangleq -\operatorname{sgn}(B_xT_{Dx} + B_yT_{Dy}) = -\operatorname{sgn}(\Delta\dot{\psi})$, use Mode 1. b) If $\operatorname{sgn}(T_z) = \operatorname{sgn}(\Delta\dot{\psi})$, use Eq. (19) with $m_x = m_y = 0$.

Mode 3. $(|\Delta\dot{\psi}| > C_{d2})$: a) If $\dot{\sigma} < C_p$, use Mode 2. b) If $\dot{\sigma} \geq C_p$, use Eqs. (19) and (21); the control is constrained to remain in this mode until $\dot{\sigma} = C_p$ is again reached.

The preceding three logic modes and the Kalman filter [Eqs. (7) and (9)] essentially represent the three-degree-of-freedom magnetic attitude control system of this study. They mechanize the suboptimal energy pointing control law [Eqs. (15)] and the spin control law [Eq. (20)].

The Skewed Coil Controller

One can simplify the mechanization of the control system by replacing the three coils by a single coil skewed (45°, for example) to the spin axis (Fig. 5). A further simplification is to use only a single axis magnetometer with its sensitive axis in the plane of the coil as indicated (rather than a 2- or 3-axis magnetometer required for the 3-coil system). For such an arrangement, if the coil current magnitude and direction are constant, an average magnetic moment is generated along the spin axis. If a constant current has its direction reversed every 180°, the average magnetic moment is in the x-y plane pointing in the direction 90° from the switch points.

The skewed coil requires the supplementation of a mechanical nutation damper if the exact damping terms of the control torque are required. However, it offers packaging advantages which are worthy of consideration.

Control System Stability

If the control gains are chosen so that the applied pointing torque is always energy optimal, the resulting system is asymptotically stable. However, for the constant gain implementation considered for this system, the applied torque (Eqs. 18a) varies directly with the magnitude of the magnetic field component B_z, and stability is not assured.

The key to the stability analysis of the primary control (Mode 1) is an extension of the generalized "circle criterion"[11] (Appendix B) in which a Lyapunov function is generated for establishing control-parameter constraints to insure stability in the time-varying magnetic field.

From Eqs. (18a), the applied pointing control torque components are $T_x = k_1(t)T_{Dx}$, and $T_y = k_1(t)T_{Dy}$, where $0 \leq k_1(t) \leq k_m$. For the control system employing only passive damping, the magnitude of $k_1(t)$ must be bounded by

$$k_m \cong Dd_p/K_{p2} \tag{22}$$

to guarantee asymptotic stability. If damping is entirely active, k_m can approach infinity. With both active and passive damping, it lies between Eq. (22) and infinity. Determination of the actual value allows one to choose control gains which insure stability in Mode 1.

The attitude stability of a system using a spin coil to provide active damping pointing control was studied in detail by Wheeler.[2] A necessary condition for stability with active damping can be shown to be

$$K_v > K_{p2}/(D + n) \tag{23}$$

by using averaging techniques. For passive damping alone, the constraint is

$$d_p > K_{OB}K_{p2}B_{1\max}{}^2/2(D + n) \tag{24}$$

Here, $B_{1\max}{}^2$ is the maximum squared value of the lateral component of the magnetic field found in the orbit. In both cases of spin-coil control, the gain K_{p1} must be set to zero.

Equations (22–24) establish necessary stability requirements based on the orbit and spacecraft parameters. The constraints are independent of the manner in which the magnetic field fluctuates.

System Performance

The pointing stability constraints of the preceding section depend upon the assumption that a constant spin speed can be successfully maintained. Control system performance in the presence of disturbance torques was analyzed by a digital computer simulation employing the exact equations of motion, the Kalman filter, the control logic, a ninth-order harmonic model of the magnetic field, and conservative values of the usual disturbance torques. Inclinations between 20° and 70° and eccentricities up to 0.7 were allowed. Perigee was held below 500 km. A typical response plot is shown in Fig. 6, where control goes from Mode 2 to Mode 3 to Mode 1. Here, the parameters used were $d_p = 0.0126$ sec^{-1}, $D = 1.6$ sec^{-1}, $C_{d1} = 0.01$ sec^{-1}, and $C_{d2} = 0.02$ sec^{-1}. For this example, the spin speed could be controlled to within 0.03 sec^{-1}, or 3% of the nominal value.

Table 1 Transient response times of controllers with and without the state estimator (Kalman filter) for various initial conditions and two values of K_{p2}

Initial conditions, rad		Position gain K_{p2}, sec^{-2}	Time-to-origin, sec	
ϕ	θ		Estimator	No estimator
0.18	0	0.03	1700	6300
0.18	0.15	0.03	1100	7300
0	0.15	0.03	700	6400
−0.18	0.15	0.03	1200	3200
0.18	0	0.1	800	700
0.18	0.15	0.1	1100	>18000
0	0.15	0.1	1100	>18000
−0.18	0.15	0.1	900	>18000

In Mode 3, the spin-speed control torque produces a pointing disturbance which is not necessarily cancelled by the pointing control from the spin coil. It is conceivable that a disturbance torque situation exists which would cause the pointing error to continue to increase in Mode 3. This situation was never observed in the simulations, however.

This attitude control system requires the presence of a state estimator from which the vehicle rates and yaw error are determined. As mentioned before, these parameters are physically observable from the roll angle because of the orbital cross-coupling [represented by the $\dot{\sigma}$ term in Eq. (3)] which exists between roll and yaw. The rate of estimating the parameters is more rapid at lower altitudes where the orbital rate is larger. Also, the estimator response time is dependent upon the magnitudes of the elements in the gain matrix $\mathbf{K}$ of the filter, which, in turn, are directly dependent upon the covariance $\mathbf{R}_1/R_2$. For large measurement noise (large R_2), estimation rates become slow. Estimation accuracy and system performance also depend upon the deviation of the actual disturbance torque and measurement noise characteristics from the white noise assumptions of Eq. (6).

Without an estimate of yaw error, one might provide a simpler pointing control using roll error measurements θ_s only, i.e.,

$$T_{Dx} = -K_{p2}\theta_s \tag{25}$$

For such a system, the body rates would be controlled by a passive nutation damper.

The Mode 1 control law [Eqs. (15)] was compared with Eq. (25) on an analog computer in both transient performance in the presence of no disturbances and steady-state performance with a variety of disturbances. Typical transient response times required to drive the spin axis to within 0.1° of the nominal from various sets of initial conditions are recorded in

Table 2 Yaw fluctuations for different torque inputs to the satellite with and without the state estimator

Type of torque[a]	Torque magnitude, 10^{-4} $N-m$		Yaw fluctuations, 10^{-2} deg	
	Yaw(x)[a]	Roll(y)[a]	Estimator	No estimator
A, D, and P	$-1\ -0.8c$	$-s$	−20 to 93	0 to 70
D,P	$-0.2c$	$-s$	− 6 to 7	−38 to 30
D only	$-0.4c$	$-0.8s$	−14 to 16	−31 to 25
P only	$0.2c$	$-0.2s$	− 7 to 9	−10 to 8

[a] A = aerodynamic, D = dipole, P = precession; $c = \cos nt$, $s = \sin nt$.

Table 1 for a system in a circular orbit with the parameters $D = 1.6 \text{ sec}^{-1}$, $n = 1.16 \times 10^{-3} \text{ sec}^{-1}$, $K_{pl} = 0$, $K_v = 0.1 \text{ sec}^{-1}$, and $I_{xx} = 10 \text{ kg-m}^2$.

The steady-state performance was evaluated using conservative approximations of the major disturbance torques which could act on a spinning satellite of cylindrical shape. Torques due to aerodynamic drag, a satellite-fixed magnetic dipole along the spin axis, and the torque required to follow the precessing orbit plane were modeled as constants and sinusoidal functions for a 16 rev/day circular orbit. Typical results of the fluctuating yaw error in systems with and without the state estimator are shown in Table 2. The same model was used as in Table 1 with $K_{p2} = 0.003 \text{ sec}^{-2}$.

In Table 1, the improvement from estimator application is evident. Rapid response is important, because it means that the average deviation from the nominal is smaller for a vehicle subjected to disturbances. From Table 2 it is seen that the magnitude of the yaw error variation is smaller with the estimator if an oscillating disturbance is primarily about the roll axis. It tends to be larger if the oscillating torque is primarily about the yaw axis. Both systems tend to have the same steady-state yaw error ϕ_c due to a constant yaw torque T_{cx} ($\phi_c \cong T_{cx}/Dn$). This ϕ_c can be decreased only by increasing the angular momentum of the symmetric spacecraft or by increasing the sensor capability to make the yaw torque observable. For the example studied, the steady-state pointing error could always be kept well below 1°.

Conclusions

A new magnetic system which provides continuous spin-axis pointing and spin-speed control for an axisymmetric spacecraft has been presented. A Kalman filter, used to estimate the spacecraft's attitude state from sampled roll-error measurements, enables the control system to provide energy optimal control, active damping, and faster transient response. A constant-coefficient, low-energy pointing control law has been algebraically determined by assuming a constant spin component of the Earth's magnetic field. Mechanization of both pointing and spin-speed control is achieved by using three logic modes based upon spin-speed variations. A single skewed coil can be used to create the magnetic control torque. Control parameter constraints have been established by a Lyapunov-function-generating technique to insure attitude stability in a time-varying magnetic field.

Full control can be provided for Earth orbits with inclinations between 20° and 70° and eccentricities up to 0.7 (perigee less than 500 km). For the example studied, the spin-speed variations can be kept less than 3% of the nominal value, and the spin axis can be maintained within 1° of normal to the orbit plane. Absolute pointing accuracy is limited by the magnitude of the yaw torque.

The orbital eccentricity has two notable effects on the control system: The resulting time-varying orbital rate $\dot{\sigma}$ has to be modeled in the Kalman filter, and the large excursions of the magnetic field—and resulting stability constraints—must be taken into account for constant-gain control laws.

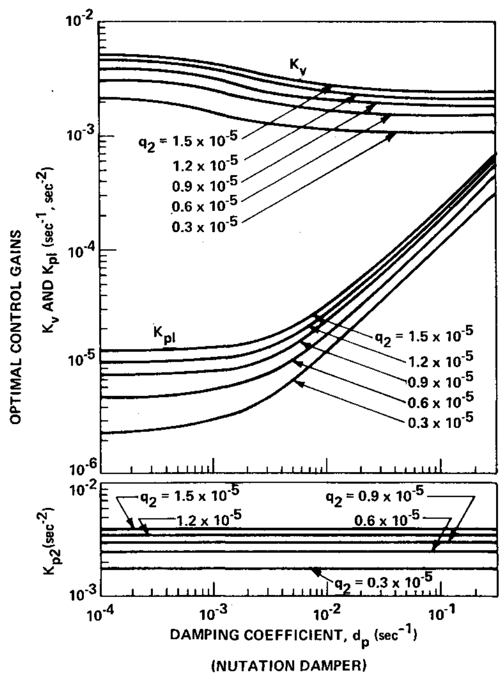

Fig. 7 Optimum gains of the control law for energy minimization of a magnetically controlled spinning satellite.

Appendix A: Algebraic Solution of the Suboptimal Energy Control Law

The effort required to solve any even-ordered quadratic matrix equations [such as Eq. (13) of the text] can be reduced considerably if the equations possess complex symmetry. The technique is now demonstrated by solving the suboptimal control law [Eq. (14)] for the system represented by Eqs. (3).

The orbital rate $\dot{\sigma}$ is temporarily treated as the constant n, and the following complex quantities are defined:

$$z \triangleq \phi + j\theta; \quad u \triangleq u_1 + ju_2$$

$$A \triangleq d_p + j(n - D); \quad B \triangleq Dn + jd_p n$$

Using these definitions reduces Eqs. (3) to the complex equations

$$\begin{bmatrix} \ddot{z} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -A & -B \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u] \tag{A1}$$

or $\dot{\mathbf{z}} = \mathbf{F}_1\mathbf{z} + \mathbf{G}_1\mathbf{u}$. Then, the matrix Eq. (13) must be placed in the Hermitian form

$$\dot{\mathbf{S}} = -\mathbf{S}\mathbf{F}_1 - \mathbf{F}_1^*\mathbf{S} - \mathbf{Q}_1 + \mathbf{S}\mathbf{G}_1\mathbf{Q}_2^{-1}\mathbf{G}_1^*\mathbf{S} \tag{A2}$$

which has the steady-state solution $\mathbf{S}_\infty$. The notation $\mathbf{F}_1^*$ refers to the complex conjugate of the transpose of $\mathbf{F}_1$.

The cost of driving the initial complex state $\mathbf{z}(t_0)$ to zero is

$$J = 0.5\mathbf{z}^*(t_0)\mathbf{S}_\infty\mathbf{z}(t_0) \tag{A3}$$

Note that

$$\mathbf{z} = \begin{bmatrix} 1 & j & 0 & 0 \\ 0 & 0 & 1 & j \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \phi \\ \theta \end{bmatrix} \triangleq \mathbf{L}\mathbf{x} \tag{A4}$$

where $\mathbf{L}$ is the indicated transformation. Let us define

$$\mathbf{P}_1 \triangleq \mathbf{L}^*\mathbf{S}_\infty\mathbf{L}, \qquad \mathbf{S}_\infty \triangleq \begin{bmatrix} s_{11} & s_{12} \\ s_{12}^* & s_{22} \end{bmatrix} \tag{A5}$$

(where s_{12}^* is the complex conjugate of s_{12}). Now, because J is a quadratic form, Eq. (A3) becomes [using Eqs. (A4) and (A5)]

$$J = 0.5\mathbf{x}^T(t_0)(\mathbf{P}_1 + \mathbf{P}_1^T)\mathbf{x}(t_0) \tag{A6}$$

Thus,

$$\mathbf{P}_\infty = 0.5(\mathbf{P}_1 + \mathbf{P}_1^T)$$

$$= \begin{bmatrix} s_{11} & 0 & Re(s_{12}) & -Im(s_{12}) \\ 0 & s_{11} & Im(s_{12}) & Re(s_{12}) \\ Re(s_{12}) & Im(s_{12}) & s_{22} & 0 \\ -Im(s_{12}) & Re(s_{12}) & 0 & s_{22} \end{bmatrix} \tag{A7}$$

This matrix only has four unknowns for which algebraic solutions can readily be found.

For the performance index

$$J = 0.5 \int_{-\infty}^{t_f} \{[\dot{\phi}^2 + \dot{\theta}^2]q_1 + [\phi^2 + \theta^2]q_2 + [u_1^2 + u_2^2]\}dt \tag{A8}$$

the matrix $\mathbf{Q}_1$ becomes

$$\mathbf{Q}_1 = \begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix}$$

and $\mathbf{Q}_2 = 1$. If one defines $s_{11} \triangleq K_v$ and $s_{12} \triangleq K_{p1} + jK_{p2}$, and sets $\dot{\mathbf{S}} = 0$, Eqs. (A2) can be placed into the form (discarding s_{22}),

$$2K_{p1}^3 + [(D + n)^2 + d_p^2 + q_1]K_{p1}^2 + 2(Dnq_1 - q_2)K_{p1} - [d_p^2q_2 + (d_pn)^2q_1 + q_1q_2] = 0 \tag{A9a}$$

$$K_v = -d_p + (d_p^2 + q_1 + 2K_{p1})^{1/2} \tag{A9b}$$

$$K_{p2} = -d_pn - [(d_pn)^2 + q_2 - K_{p1}^2 - 2DnK_{p1}]^{1/2} \tag{A9c}$$

The solutions to (A9) for K_v, K_{p1}, and K_{p2} represent the gains for the control law of Eqs. (15) and can rapidly be obtained. The choices of values for q_1 and q_2 in Eq. (A8) depend upon the desired system response speed (the type of control available); q_1 and q_2 are used to penalize appropriately the attitude errors and rates in driving the error state to zero.

As an example of this method, solutions to Eqs. (A9) were found for a spacecraft with the parameter $D = 1.6$ sec^{-1}. Values of the damping coefficients d_p ranged from 10^{-4} to 10^{-1} sec^{-1}. The mean orbital rate n was set to values of 1.09×10^{-3} sec^{-1} and 2.18×10^{-4} sec^{-1}, and q_1 was set to zero (which allows unlimited rates to reach the origin) while q_2 was given five values which produce control gains suitable for a magnetic control system. The results are delineated in Fig. 7. The differences due to variations of n are undistinguishable.

Appendix B: Mode 1 Stability Constraints

The necessary condition for Mode 1 stability can be obtained by an extension of the generalized "circle criterion." A Lyapunov function is generated which establishes the bounds on acceptable excursions of the magnetic field component B_z. The circle criterion is essentially formulated in the following theorem: Consider the system with state equations

$$\dot{\mathbf{x}} = \mathbf{Fx} + \mathbf{Gu} = [\mathbf{F} - \mathbf{GK}_g(t)\mathbf{H}]\mathbf{x}$$
$$\mathbf{y} = \mathbf{Hx} \tag{B1}$$

where $\mathbf{u}$ is governed by the time-varying gain matrix $\mathbf{K}_g(t)$, and is expressed as $\mathbf{u} = -\mathbf{K}_g(t)\mathbf{y}(t)$. The matrices $(\mathbf{F,G,H})$ are constant. The matrix $\mathbf{K}_g(t) = \text{diag}\,\{k_1(t), \ldots, k_p(t)\}$ satisfies the condition $\mathbf{K}_1 \leq \mathbf{K}_g(t) \leq \mathbf{K}_2$ for all $t > 0$, where $\mathbf{K}_1 = \text{diag}\,\{k_{11}, \ldots, k_{1p}\}$ and $\mathbf{K}_2 = \text{diag}\,\{k_{21}, \ldots, k_{2p}\}$ are constant and $\mathbf{K}_2 - \mathbf{K}_1 > 0$, i.e., positive definite. Now, if a bounded, symmetric, positive definite matrix $\mathbf{M}_r$ can be found which satisfies the equations

$$\mathbf{M}_r(\mathbf{F} - \mu\mathbf{I} - \mathbf{GK}_1\mathbf{H}) + (\mathbf{F} - \mu\mathbf{I} - \mathbf{GK}_1\mathbf{H})^T\mathbf{M}_r = -\mathbf{N}_r\mathbf{N}_r^T \tag{B2a}$$

$$\mathbf{M}_r\mathbf{G} = \mathbf{H}^T/2 - \mathbf{N}_r\mathbf{P}_r;\ \mathbf{P}_r^T\mathbf{P}_r = (\mathbf{K}_2 - \mathbf{K}_1)^{-1} \tag{B2b}$$

where $\mu \leq 0$, then the system is stable in the large. If $\mu < 0$, the system is asymptotically stable in the large.

The proof of this theorem involves showing that $\mathbf{x}^T\mathbf{M}_r\mathbf{x}$ is a suitable Lyapunov function. Establishment of system stability requires solving Eqs. (B2) for a bounded symmetric positive definite $\mathbf{M}_r$.

Equations (B2) can be combined to yield

$$\mathbf{M}_r[\mathbf{F} - \mu\mathbf{I} - \mathbf{G}(\mathbf{K}_1 + \mathbf{K}_2)\mathbf{H}/2] + [\mathbf{F} - \mu\mathbf{I} - \mathbf{G}(\mathbf{K}_1 + \mathbf{K}_2)\mathbf{H}/2]^T\mathbf{M}_r + \mathbf{M}_r\mathbf{G}(\mathbf{K}_2 - \mathbf{K}_1)\mathbf{G}^T\mathbf{M}_r + \mathbf{H}^T(\mathbf{K}_2 - \mathbf{K}_1)\mathbf{H}/4 = 0 \tag{B3}$$

which is a matrix quadratic equation. This theorem has direct application to the Mode 1 controller with its time-varying gains. The proof of stability requires finding an $\mathbf{M}_r$ which satisfies (B3). Again, use can be made of the complex symmetry property of the equations to provide means of obtaining a rapid algebraic solution.

Consider first the skewed coil controller having only passive nutation damping. From Eqs. (15) and (18), the actual magnetic pointing control components are

$$T_x = -k_1(t)(K_{p1}\phi + K_{p2}\theta)$$
$$T_y = -k_1(t)(-K_{p2}\phi + K_{p1}\theta)$$

where $k_1(t) = K_{Bz}B_z(t)$. If a circular orbit is assumed, and $0 \leq k_1(t) \leq k_m$, where k_m is a constant, then two matrices used in (B3) are

$$\mathbf{GK}_2\mathbf{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_m & 0 \\ 0 & k_m \end{bmatrix} \begin{bmatrix} 0 & 0 & K_{p1} & K_{p2} \\ 0 & 0 & -K_{p2} & K_{p1} \end{bmatrix}$$

and $\mathbf{K}_1 = 0$. $\mathbf{M}_r$ has the form

$$\mathbf{M}_r = \begin{bmatrix} m_1 & 0 & m_2 & m_3 \\ 0 & m_1 & -m_3 & m_2 \\ m_2 & -m_3 & m_4 & 0 \\ m_3 & m_2 & 0 & m_4 \end{bmatrix}$$

and is positive definite if

$$m_1 > 0, \text{ and } m_1m_4 > m_2^2 + m_3^2 \tag{B4}$$

A systematic way thus is established for determining the limit of position gain variation that assures stability in Mode 1 control due to time variations of the magnetic field. The constant k_m is raised in Eqs. (B3) until a solution can no longer be found which satisfies Eqs. (B4). The approximate value of k_m can be shown to be

$$k_m \cong Dd_p/K_{p2} \tag{B5}$$

Equation (B5) determines the damper size needed to supplement the skewed coil controller.

For a satellite actively damped with optimal gains,

$$\mathbf{H} = \begin{bmatrix} K_v & 0 & K_{p1} & K_{p2} \\ 0 & K_v & -K_{p2} & K_{p1} \end{bmatrix}$$

and no damper term d_p is present in the matrix $\mathbf{F}$. This system is always stable if the lower bound matrix $\mathbf{K}_1$ is nonnegative. The upper bound k_m can approach infinity. The technique can also be applied to systems with both active and passive damping.

The orbital rate $\dot{\sigma}$ can also be treated as a time-varying gain. This adds some complication to the solution of Eqs. (B3). However, examples have demonstrated that the slow variation of $\dot{\sigma}$ between perigee and apogee values has an inconsequential affect on the stability constraints. Eccentricity, of course, does affect the magnitude of excursions of B_z and the constant k_m.

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Semipassive and Active Nutation Dampers for Dual-Spin Spacecraft

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The acquisition of flight data from orbiting dual-spin spacecraft has emphasized the difficulty of providing assurance that dissipative effects tending to reduce nutation outweigh those tending to increase it. Although passive platform-mounted devices for attenuating nutation are generally reliable and conceptually simple, their effectiveness may be limited. This paper is concerned with alternative devices, namely, single-axis control moment gyros (CMG's) whose rotational motion relative to the spacecraft is either a) restrained passively by a spring and dashpot, or b) controlled actively by a torque motor driven with sensed information from a rectilinear accelerometer. Although the passively controlled device can operate effectively only when placed on the platform, the active device can be effective when placed on the rotor as well. Both devices are capable of reducing nutation several times faster than passive dampers of equal mass.

Nomenclature

$\mathbf{a}^P$ = acceleration of P, Eq. (32)
a_m = measured component of $\mathbf{a}^P$, Eq. (33)
B,B' = two primary bodies of spacecraft
c = CMG gimbal damping constant
$\mathbf{d}$ = $\sum_{i=1}^{3} d_i\mathbf{x}_i$, position vector from system cm to CMG cm
e_a = armature voltage
f = tuning factor, Eq. (31)
$\mathbf{h}_g$ = CMG angular momentum
h = magnitude of $\mathbf{h}_g$, Eq. (29)
i_a = armature current
I_i = moment of inertia of B plus B' for X_i
I_3' = moment of inertia of B' for X'_3
$[\bar{I}_1,\bar{I}_2,\bar{I}_3]$ = $[I_1 + m(d_2^2 + d_3^2), I_2 + m(d_1^2 + d_3^2), I_3 + m(d_1^2 + d_2^2)]$
J_1,J_3 = centroidal principal moments of inertia of CMG wheel for transverse and symmetry axes, respectively
J_m = motor rotor moment of inertia
K = combined gain of accelerometer and amplifier
K_e = back emf of motor
K_T = torque constant of motor
k = semipassive CMG spring constant
m = mass of CMG
$\mathbf{M}_t$ = $\sum_{i=1}^{2} M_i\mathbf{x}_i$, transverse reaction torque applied to B by CMG
p = passive CMG natural frequency, Eq. (9b)
$\mathbf{p}$ = $\sum_{i=1}^{3} p_i\mathbf{x}_i$, position vector from system cm to accelerometer
P = power consumed by CMG torque motor
P_{ave} = average value of P
R_a = armature resistance
r = ratio of transverse inertias, Eq. (3c)
$\mathbf{T}$ = $\sum_{i=1}^{3} T_i\mathbf{x}_i$, torque applied to CMG
$\mathbf{x}_i$ = unit vector parallel to X_i
X_1,X_2,X_3 = reference axes fixed in B and passing through system cm
X_3' = symmetry axis of B'
β = motor friction constant combining effects of viscous friction and back emf
γ = gimbal angle, Fig. 1
γ_0 = steady-state amplitude of γ, Eqs. (15) and (42)
Δ = measure of gyro size, Eq. (3c)
ζ = passive CMG damping ratio, Eq. (9b)
η,ν = Eq. (3d) and (9a), respectively
λ_1,λ_2 = Eqs. (3a,b)

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